THEORY OF ELECTROSTATIC FOCUSING OF INTENSE BEAMS OF CHARGED PARTICLES

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The mathematical formulation of the problem of determining the electrodes for the formation of intense beams of charged particles reduces to the solution of the Cauchy problem for the Laplace equation. One can proceed either by separating the variables [1] or on the basis of the theory of analytic continuation [2–5]. This approach can be used for plane or axisymmetric flows. An algorithm for the construction of the analytic solution, which can also be used in the three-dimensional case, is given below. It is assumed that the beam boundary coincides with the coordinate surface $x^1 = 0$ of an orthogonal system x^i $(i = 1, 2, 3)$. The solution is put in the form of a series in x^1 with coefficients dependent on x^2 and x^3, determined from recurrence relations. The case of emission limited by space charge and temperature generally gives rise to difficulties due to the divergence of the series which makes it impossible to calculate the zero equipotential by the indicated method.

As an example, the formation of beams with an elliptic cross section is considered in the following cases: (1) periodic variation of the z-component of the velocity; (2) nonmonotonic variation of the potential in one-dimensional flow between planes z = const; (3) a beam accelerated in accordance with a 3/2 law.

In the construction of the expansions the conditions on the boundary are satisfied exactly by the first two terms of the series.

1. CONSTRUCTION OF GENERAL SOLUTION

We relate the surface of the beam to an orthogonal* coordinate system x^i $(i = 1, 2, 3)$ and take $x^1 = 0$ as its equation in this system. The problem consists in constructing the solution of the Laplace equation

$$\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ik}\frac{\partial\varphi}{\partial x^k}\right) = 0 \qquad (1.1)$$

satisfying the following conditions when $x^1 = 0$:

$$\varphi = \Phi(x^2,\ x^3), \qquad \sqrt{g^{11}}\,\partial\varphi/\partial x^1 = \varepsilon(x^2,\ x^3). \qquad (1.2)$$

Since it is intended to seek the solution in the form of a series in x^1 with coefficients dependent on x^2 and x^3

$$\varphi = \varphi_k (x^1)^k \qquad (k = 0, 1, \ldots), \qquad (1.3)$$

we put the elements of the metric tensor g_{ik}, and also of the combination $(g)^{1/2}g^{ik}$, in the form of similar series

$$g_{11} = a_k (x^1)^k, \quad \sqrt{g}\,g^{11} = \alpha_k (x^1)^k,$$

$$\sqrt{g}\,g^{22} = \beta_k (x^1)^k, \quad \sqrt{g}\,g^{33} = \gamma_k (x^1)^k$$

$$(k = 0, 1, \ldots). \qquad (1.4)$$

Here the subscripts k have the usual meaning of the ordinal number of the terms of the series and powers.

*The system x^i is assumed orthogonal for simplicity; nonorthogonality only complicates the calculations.

Substituting (1.3) and (1.4) into (1.1) and equating the coefficients of equal powers of x^1 we arrive at the recurrence relations for the determination of φ_k

$$s\sum_{k=0}^{s}(s-k+1)\,\alpha_k\varphi_{s-k+1} +$$

$$+\sum_{k=0}^{s-1}\{[\beta_k(\varphi_{s-k-1})_2']_2' + [\gamma_k(\varphi_{s-k-1})_3']_3'\} = 0$$

$$(s = 1, 2, \ldots). \qquad (1.5)$$

Here the prime denotes differentiation and the subscript indicates the coordinate with respect to which differentiation is carried out.

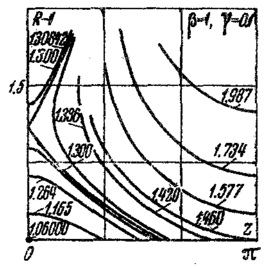

Fig. 1

We write out a few terms of the expansion of the potential

$$\varphi_0 = \Phi, \quad \varphi_1 = \varepsilon a_0^{1/2},$$

$$\varphi_2 = [({}^1/_4\, a_1/a_0^{3/2} + {}^1/_2\, T_1)\,\varepsilon + {}^1/_2\, T_2\Phi_P' +$$

$$+ {}^1/_2\, T_3\Phi_Q' - {}^1/_2(\Phi_P'' + \Phi_Q'')]\,a_0,$$

$$\varphi_3 = \{{}^1/_6\,\varepsilon a_2/a_0^2 - {}^1/_{24}\,\varepsilon a_1^2/a_0^3 + {}^1/_4(a_1/a_0^{3/2})(T_1\varepsilon +$$

$$+ T_2\Phi_P' + T_3\Phi_Q' - \Phi_P'' - \Phi_Q'') + {}^1/_6\,\varepsilon T_{1S}' +$$

$$+ {}^1/_3\, T_1({}^1/_2\, T_1\varepsilon + T_2\Phi_P' + T_3\Phi_Q' - \Phi_P'' - \Phi_Q'') +$$

$$+ {}^1/_6[(2k_1 + k_2)(\varepsilon_P' - k_1\varepsilon) + (2\delta_1 + \delta_2)(\varepsilon_Q' - \delta_1\varepsilon) -$$

$$-(\varepsilon_P' - k_1\varepsilon)_P' - (\varepsilon_Q' - \delta_1\varepsilon)_Q'] + {}^1/_6[\varkappa_1 T_2 -$$

$$-\varkappa_2(2k_1 + k_2) - \varkappa_{1P}' + \varkappa_{2P}' + k_{1S}']\,\Phi_P' +$$

$$+ {}^1/_6[\varkappa_2 T_3 - \varkappa_1(2\delta_1 + \delta_2) + \varkappa_{1Q}' - \varkappa_{2Q}' + \delta_{1S}']\,\Phi_Q' -$$

$$- {}^1/_6(\varkappa_1 - \varkappa_2)(\Phi_P'' - \Phi_Q'')\}\,a_0^{3/2}. \qquad (1.6)$$

Beginning with φ_2 the coefficients in (1.3) depend not only on the derivatives of the functions Φ and ε with respect to the arcs P and Q of the curvilinear axes x^2 and x^3

$$P = \int\sqrt{g_{22}}\,dx^2, \qquad Q = \int\sqrt{g_{33}}\,dx^3,$$

but also on the geometry of the surface. Here $\varkappa_1$ and $\varkappa_2$ are the principal curvatures of the surface $x^1 = 0$;

$T_1 = \varkappa_1 + \varkappa_2$ is its total curvature; k_1, k_2 and T_2, and δ_1, δ_2 and T_3 are the principal and total curvatures of the

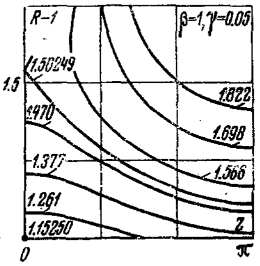

Fig. 2

coordinate surfaces x^2 = const and x^3 = const, calculated for $x^1 = 0$:

$$\varkappa_1 = -\frac{1}{2\sqrt{g_{11}}}\frac{\partial \ln g_{22}}{\partial x^1}, \qquad \varkappa_2 = -\frac{1}{2\sqrt{g_{11}}}\frac{\partial \ln g_{33}}{\partial x^1},$$

$$k_1 = -\frac{1}{2\sqrt{g_{22}}}\frac{\partial \ln g_{11}}{\partial x^2}, \qquad k_2 = -\frac{1}{2\sqrt{g_{22}}}\frac{\partial \ln g_{33}}{\partial x^2},$$

$$\delta_1 = -\frac{1}{2\sqrt{g_{33}}}\frac{\partial \ln g_{11}}{\partial x^3}, \qquad \delta_2 = -\frac{1}{2\sqrt{g_{33}}}\frac{\partial \ln g_{22}}{\partial x^3}.$$

In addition, S is the length of the arc of the curvilinear axis x^1, orthogonal to the beam surface

$$S = \int \sqrt{g_{11}}\, dx^1,$$

so that T'_{1S}, for instance, means the rate of change of the total curvature of the surfaces x^1 = const, calculated for $x^1 = 0$. The value of the coefficients depends not only on the geometry of the problem, but also on the meaning of the parameter in terms of which the expansion is made. We can thus construct solutions in systems obtained from one another by transformations

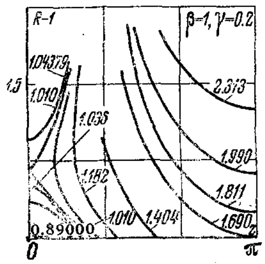

Fig. 3

of the type $(x^1)' = f(x^1)$, which do not alter the geometry of the coordinate surfaces x^1 = const; the elements of the metric tensor, however, become different. For instance, for a cylindrical beam R = const in the case of treatment in polar coordinates (1) R, ψ, z and coordinates (2) ln R, ψ, z we have

$$(1) \quad g_{11} = 1, \qquad g_{22} = (x^1)^2,$$

$$a_0 = 1, \qquad a_2 = a_3 = \ldots = 0,$$

$$(2) \quad g_{11} = \exp(2x^1), \quad g_{22} = \exp(2x^1), \quad a_k = 2^k/k!.$$

Transformations $(x^1)' = f(x^1)$ can be useful for simplifying the formulas for determining φ_k in (1.3); in addition, the region of convergence of (1.3) depends on the meaning of x^1.

Expressions (1.6) take a universal form if we convert to an expansion in terms of the arc S of the curvilinear axis x^1 by means of the relationship

$$s = a_0^{1/2} x^1 = S - \frac{1}{4}\frac{a_1}{a_0^{3/2}} S^2 + \frac{1}{6}\left(\frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}\right) S^3 + \ldots.$$

Performing this transformation, we obtain

$$\begin{aligned} \varphi = \Phi + \varepsilon S + {}^1/_2 (T_1\varepsilon + T_2\Phi_P' + \\ + T_3\Phi_Q' - \Phi_P'' - \Phi_Q'') S^2 + {}^1/_6 \{2T_1 ({}^1/_2 T_1\varepsilon + \\ + T_2\Phi_P' + T_3\Phi_Q' - \Phi_P'' - \Phi_Q'') + \varepsilon T_{1S}' + \\ + (2k_1 + k_2)(\varepsilon_P' - k_1\varepsilon) + (2\delta_1 + \delta_2)(\varepsilon_Q' - \delta_1\varepsilon) - \\ - (\varepsilon_P' - k_1\varepsilon)_P' - (\varepsilon_Q' - \delta_1\varepsilon)_Q' + \\ + [\varkappa_1 T_2 - \varkappa_2 (2k_1 + k_2) - \varkappa_{1P}' + \varkappa_{2P}' + k_{1S}'] \Phi_P' + \\ + [\varkappa_2 T_3 - \varkappa_1 (2\delta_1 + \delta_2) + \varkappa_{1Q}' - \varkappa_{2Q}' + \delta_{1S}'] \Phi_Q' - \\ - (\varkappa_1 - \varkappa_2)(\Phi_P'' - \Phi_Q'')\} S^3 + \ldots. \end{aligned} \tag{1.7}$$

2. SOME COMMENTS

The proposed algorithm for construction of the solution enables us to consider three-dimensional problems. In addition, in the general case the question of

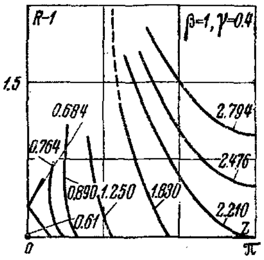

Fig. 4

convergence of expression (1.3) remains open. Below we will take specific examples to demonstrate the difficulties which can be encountered in the formal application of the method and we also attempt to define its range of validity.

2.1. Formation of ribbon-shaped beam. We consider the problem of formation of a ribbon-shaped beam in the case of emission limited by space charge. This, as we know [6], has an analytical solution expressed by the formulas

$$\varphi = (x^2 + y^2)^{2/3} \cos {}^4/_3 \operatorname{arctg}(y/x) = R^{4/3} \cos {}^4/_3 \psi. \tag{2.1}$$

The construction of expansion (1.3) in Cartesian coordinates leads to the following result:

$$s^2\varphi_{s+1} + \varphi_{s-1}'' = 0,$$

$$\varphi = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\frac{d^{2k}\Phi}{dx^{2k}} y^{2k}, \quad \Phi = x^{4/3}. \tag{2.2}$$

It is evident that (2.2) converges when $|y/x| < 1$, although expression (2.1) has a meaning for any x, y. Formula (2.2) is valid for the construction of surfaces φ = const sufficiently far from the emitter, but does not contribute anything to the determination of the zero equipotential.

It is of interest that on conversion to polar coordinates $x^1 = \psi$, $x^2 = R$, we have

$$g_{11} = R^2, \quad g_{22} = 1, \quad \alpha_0 = R^{-1}, \quad \beta_0 = R,$$

$$\alpha_k = \beta_k = 0 \quad (k > 0),$$

$$s(s+1) R^{-1} \varphi_{s+1} + (R\varphi'_{s-1})' = 0,$$

$$\Phi = R^{4/3}, \quad \varepsilon = 0. \qquad (2.3)$$

It is easy to see that all $\varphi_{2k+1} = 0$, $\varphi_{2k} = c_{2k} R^{4/3}$, where c_{2k} are the coefficients of the expansion of $\cos 4\psi/3$ in a series convergent for all ψ.

The same difficulty arises in the problem of formation of a cylindrical beam [7] accelerated in accordance with a 3/2 law. The difficulty can be overcome in exactly the same way, by converting from the system $x^1 = R$, $x^2 = z$, $x^3 = \psi$ to coordinates

$$x^1 = [(R-1)^2 + z^2]^{1/2},$$

$$x^2 = \operatorname{arc\,tg}\,[(R-1)/z], \quad x^3 = \psi.$$

Thus, in the treatment of flows with special points on the beam boundary (for instance, emission limited by space charge and temperature), we cannot, generally speaking, obtain a complete picture

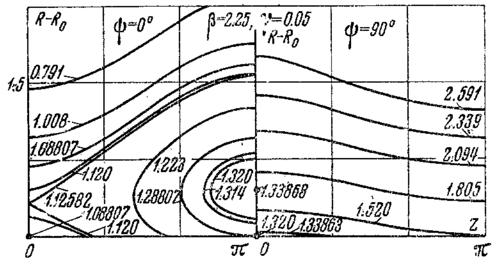

Fig. 5

of the equipotential surfaces without special contrivances. In the opposite case (nonzero velocity everywhere) we can expect that the region and rate of convergence will depend only on the geometry of the boundary, the distance from it, and the form of the functions Φ and ε.

2.2. **Formation of annular beam.** Using the method of §1, we determine the focusing electrodes for the annular electrostatic beam considered in [8]. The particles move in circular paths, and the potential in the region filled by charges is given by the expression

$$\varphi = R^{-2} (\sin^3/_2\,\psi)^{4/3}.$$

Let $x^1 = \ln R$, $x^2 = \psi$. Then

$$g_{11} = g_{22} = \sqrt{g} = \exp(2x^1),$$

$$\sqrt{g}g^{11} = \sqrt{g}g^{22} = 1, \quad \alpha_0 = \beta_0 = 1,$$

$$\alpha_k = \beta_k = 0 \quad (k > 0).$$

Relationships (1.5) take the form

$$s(s+1)\varphi_{s+1} + \varphi''_{s-1} = 0.$$

The conditions on the beam boundary R = 1 in this case are

$$\varphi_0 = \Phi = (\sin {}^3/_2\,\psi)^{4/3}, \qquad \varphi_1 = -2\Phi.$$

The solution of the problem is given by the series

$$\varphi_{2k} = \frac{(-1)^k}{(2k)!}\Phi^{(2k)}, \qquad \varphi_{2k+1} = \frac{2(-1)^{k+1}}{(2k+1)!}\Phi^{(2k)},$$

$$\varphi = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{d^{2k}\Phi}{d\psi^{2k}} \Big(1 - \frac{2}{2k+1}\ln R\Big)(\ln R)^{2k}, \qquad (2.4)$$

which converges absolutely for all R and $\psi \neq 2\pi k/3$ $(k = 0, 1, 2)$. Expression (2.4) can be used to construct the complete set of surfaces φ = const and is another new form of the solution [1, 2].

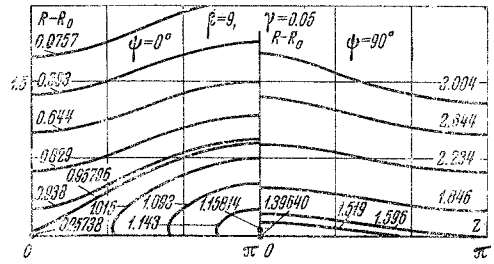

Fig. 6

2.3. **Periodic focusing of ribbon-shaped beam.** In the theory of periodic focusing there are approximate solutions of the formation problem based on approximation of the nonmonotonic variation of potential on the beam boundary by a quadratic parabola or a cosine curve

$$\varphi_0 = \alpha + (1-\alpha)\,(x/\sigma - 1)^2,$$

$$\varphi_0 = 1 - (1-\alpha)\cos(\pi x/2\sigma), \quad \varepsilon = 0. \qquad (2.5)$$

Here α and σ are constants determining the minimum value of the potential and the position of the minimum. The use of relationships (1.5) in Cartesian coordinates leads to expressions (2.2) with Φ given by formulas (2.5). As a result we obtain

$$\varphi = \alpha + (1-\alpha)\,[(x/\sigma - 1)^2 - y^2],$$

$$\varphi = 1 - (1-\alpha)\cos\frac{\pi x}{2\sigma}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\Big(\frac{\pi y}{2\sigma}\Big)^{2k} =$$

$$= 1 - (1-\alpha)\cos\frac{\pi x}{2\sigma}\operatorname{ch}\frac{\pi y}{2\sigma}.$$

Another example, where series (1.3) can be summed, thus reducing them to elementary functions, will be given below.

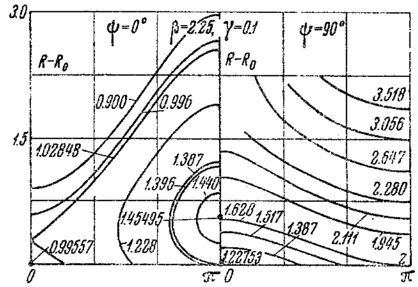

Fig. 7

3. FORMATION OF BEAMS OF ELLIPTIC CROSS SECTION

As an illustration of the general method we give the solution of several problems of formation of beams of elliptic cross section. We will use elliptic cylindrical coordinates ξ, η, z, connected with the Cartesian coordinates x, y, z by the formulas

$$x = a\sqrt{\beta - 1}\,\mathrm{sh}\,\xi \sin\eta,$$

$$y = a\sqrt{\beta - 1}\,\mathrm{ch}\,\xi \cos\eta, \quad z = z. \tag{3.1}$$

Let $\xi_0 \leq \xi < \infty$, $0 \leq \eta \leq 2$ be the Laplace region, and let a be the semiminor axis of the ellipse-boundary, $\xi = \xi_0$, $\beta = (b/a)^2$. The

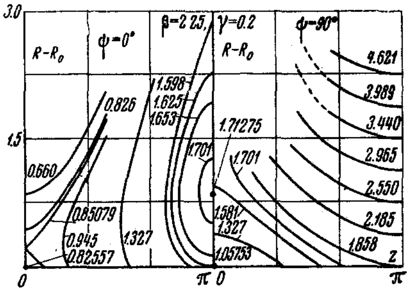

Fig. 8

metric in (3.1) is given by the expressions

$$g_{11} = g_{22} = \sqrt{g} = a^2(\beta - 1)(\mathrm{sh}^2\xi + \sin^2\eta), \qquad g_{33} = 1. \tag{3.2}$$

Then $\alpha_0 = 1$, $\beta_0 = 1$, $\alpha_k = \beta_k = 0$ when $k > 0$ and

$$\gamma_0 = \gamma_0(\eta) = a^2(\beta - 1)(\mathrm{sh}^2\xi_0 + \sin^2\eta),$$

$$\gamma_{2k-1} = a^2(\beta - 1)\,\mathrm{sh}\,2\xi_0 \frac{2^{2k-2}}{(2k-1)!} = \mathrm{const},$$

$$\gamma_{2k} = a^2(\beta - 1)\,\mathrm{ch}\,2\xi_0 \frac{2^{2k-1}}{(2k)!} = \mathrm{const}.$$

The recurrence relations (1.5) take the form

$$s(s+1)\varphi_{s+1} + (\varphi_{s-1})_\eta'' + \sum_{k=0}^{s-1} \gamma_k (\varphi_{s-k-1})_z'' = 0 \quad (s = 1, 2, \ldots). \tag{3.3}$$

Taking $\mathrm{th}^2\xi_0 = 1/\beta$, we arrive at the final formulas for γ_k

$$\gamma_0 = a^2 H(\eta) = {}^1/_2\, a^2 [(\beta + 1) - (\beta - 1)\cos 2\eta],$$

$$\gamma_{2k-1} = a^2\sqrt{\beta}\,\frac{2^{2k-1}}{(2k-1)!}, \qquad \gamma_{2k} = a^2(\beta + 1)\frac{2^{2k-1}}{(2k)!}. \tag{3.4}$$

We will assume henceforth that the potential on the beam boundary consists of the sum of functions $P(\eta)$ and $W(z)$, and the normal derivative depends only on η.

3.1. **Coefficients of expansion in the general case.** Using the recurrence relations (3.3) and confining ourselves to the first twelve terms of the expansion, we obtain

$$\varphi_0 = P(\eta) + W(z), \qquad \varphi_1 = \varepsilon(\eta),$$

$$\varphi_2 = -\frac{1}{2!}P'' - \frac{1}{2!}a^2 H(\eta) W'',$$

$$\varphi_3 = -\frac{1}{3!}\varepsilon'' - \frac{1}{3} abW'',$$

$$\varphi_4 = \frac{1}{4!}P^{\mathrm{IV}} + \frac{1}{4!}a^4 H^2(\eta) W^{\mathrm{IV}} - \frac{1}{6}a^2 H(\eta) W'',$$

$$\varphi_5 = \frac{1}{5!}\varepsilon^{\mathrm{IV}} + \frac{1}{15}a^3 b H(\eta) W^{\mathrm{IV}} - \frac{1}{15}abW'',$$

$$\varphi_6 = -\frac{1}{6!}P^{\mathrm{VI}} - \frac{1}{6!}a^6 H^3(\eta) W^{\mathrm{VI}} + \frac{1}{36}a^4\left[H^2(\eta) + \frac{6}{5}\beta\right] W^{\mathrm{IV}} - \frac{1}{45}a^2 H(\eta) W'',$$

$$\varphi_7 = -\frac{1}{7!}\varepsilon^{\mathrm{VI}} - \frac{1}{280}a^5 b H^2(\eta) W^{\mathrm{VI}} + \frac{2}{63}a^3 b\left[H(\eta) + \frac{3}{20}(\beta + 1)\right] W^{\mathrm{IV}} - \frac{2}{315}abW'',$$

$$\varphi_8 = \frac{1}{8!}P^{\mathrm{VIII}} + \frac{1}{8!}a^8 H^4(\eta) W^{\mathrm{VIII}} - \frac{1}{720}a^6 H(\eta)\left[H^2(\eta) + \frac{18}{7}\beta\right] W^{\mathrm{VI}} + \frac{1}{120}a^4\left[H^2(\eta) + \frac{12}{7}\beta\right] W^{\mathrm{IV}} - \frac{1}{630}a^2 H(\eta) W'',$$

$$\varphi_9 = \frac{1}{9!}\varepsilon^{\mathrm{VIII}} + \frac{1}{11340}a^7 b H^3(\eta) W^{\mathrm{VIII}} - \frac{1}{360}a^5 b\left[H^2(\eta) + \frac{\beta + 1}{7}H(\eta) + \frac{10}{21}\beta\right] W^{\mathrm{VI}} + \frac{1}{135}a^3 b\left[H(\eta) + \frac{3}{14}(\beta + 1)\right] W^{\mathrm{IV}} - \frac{1}{2835}abW'',$$

$$\varphi_{10} = -\frac{1}{10!}P^{\mathrm{X}} - \frac{1}{10!}a^{10} H^5(\eta) W^{\mathrm{X}} + \frac{1}{30240}a^8 H^2(\eta)[H^2(\eta) + 4\beta] W^{\mathrm{VIII}} - \frac{7}{10800}a^6\left[H^3(\eta) - \frac{180}{49}\beta H(\eta) + \frac{30}{7}\beta(\beta + 1)\right] W^{\mathrm{VI}} + \frac{17}{11340}a^4\left[H^2(\eta) + \frac{162}{85}\beta\right] W^{\mathrm{IV}} - \frac{1}{14175}a^2 H(\eta) W'',$$

$$\varphi_{11} = -\frac{1}{11!}\varepsilon^{\mathrm{X}} - \frac{1}{798336}a^9 b H^4(\eta) W^{\mathrm{X}} + \frac{1}{10395}a^7 b H(\eta)\left[H^2(\eta) + \frac{\beta + 1}{8}H(\eta) + \beta\right] W^{\mathrm{VIII}} - \frac{1}{110}a^5 b\left[\frac{7}{60}H^2(\eta) + \frac{\beta + 1}{42}H(\eta) + \frac{1}{252}(\beta^2 + 26\beta + 1)\right] W^{\mathrm{VI}} + \frac{1}{110}a^3 b\left[\frac{68}{567}H(\eta) + \frac{\beta + 1}{35}\right] W^{\mathrm{IV}} - \frac{2}{155925}abW''. \tag{3.5}$$

We note that the coefficients of the potential expansion for a cylindrical beam are obtained from (3.5) with $a = b = \beta = H = 1$. Henceforth, as a characteristic length in the plane ξ, η we take the semiminor axis of the ellipse-boundary; for this we put $a = 1$ in formulas (3.5).

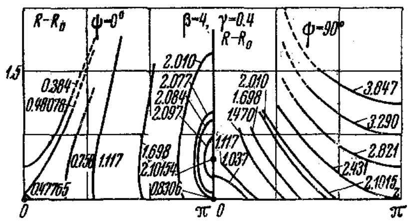

Fig. 9

3.2. **Beam with periodic variation of z-component of velocity.** We determine the shaping electrodes for a flow with periodical variation of the z-component of the velocity, the case investigated in [9]. The particles in this case move over the surface of elliptic cylinders ξ = const in a homogeneous magnetic field $H = H_z$. The conditions at the beam boundary are given by the formulas

$$\varphi_0 = \frac{\alpha}{2}\frac{\beta + 1}{\beta} - \frac{\alpha}{2}\frac{\beta - 1}{\beta}\cos 2\eta + W(z),$$

$$\alpha = \frac{\beta^2}{1 + \beta^2}\left(\frac{a}{d}\right)^2, \quad \beta = \frac{\omega_L + \Omega}{\omega_L - \Omega},$$

$$\varphi_1 = \frac{\alpha(\beta^2 + 1)}{\beta^{3/2}}\left(1 - \frac{\beta^2 - 1}{\beta^2 + 1}\cos 2\eta\right),$$

$$d = \frac{\sqrt{2}\,\pi e J_0}{m\omega^3}, \quad \omega^2 = \omega_L^2 + \Omega^2. \tag{3.6}$$

Here e/m is the specific charge of the particle, J_0 is the current density in the z-direction, ω_L is the Larmor frequency, Ω = const, and $\omega_L > \Omega$. Formulas (3.6) are written in the dimensionless variables (the symbol for the dimensionless quantity is omitted) which will be used in the subsequent treatment,

$$x^\circ = \frac{x}{a}, \quad y^\circ = \frac{y}{a}, \quad z^\circ = \frac{z}{d}, \quad \varphi^\circ = -\frac{e\varphi}{m\omega^2 d^2}. \tag{3.7}$$

The function $W = W(z)$ is assigned parametrically

$$W = (1 - \gamma \cos t)^2, \qquad z = t - \gamma \sin t. \tag{3.8}$$

Some of the terms of the series for φ, containing $P(\eta)$, $\varepsilon(\eta)$, and their derivatives, can be summed. We then obtain

$$\varphi = \alpha\left[\frac{\beta+1}{2\beta} + \frac{\beta^2+1}{\beta^{3/2}}\Xi - \frac{\beta-1}{2\beta}\cos 2\eta\left(\operatorname{ch} 2\Xi + \frac{\beta+1}{\beta^{1/2}}\operatorname{sh} 2\Xi\right)\right] + W + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}H^k W^{(2k)}\Xi^{2k} - \frac{1}{3}\sqrt{\beta}W''\Xi^3 -$$
$$- \frac{1}{6}HW''\Xi^4 + \frac{1}{15}\sqrt{\beta}(HW^{IV} - W'')\Xi^5 + \left[\frac{1}{36}\left(H^2 + \frac{6}{5}\beta\right)W^{IV} - \frac{1}{45}HW''\right]\Xi^6 +$$
$$+ \sqrt{\beta}\left\{-\frac{1}{280}H^2W^{VI} + \frac{2}{63}\left[H + \frac{3}{20}(\beta+1)\right]W^{IV} - \frac{2}{315}W''\right\}\Xi^7 + \left[-\frac{1}{720}H\left(H^2 + \frac{18}{7}\beta\right)W^{VI} + \frac{1}{120}\left(H^2 + \frac{12}{7}\beta\right)W^{IV} - \frac{1}{630}HW''\right]\Xi^8 +$$
$$+ \sqrt{\beta}\left\{\frac{1}{11340}H^3W^{VIII} - \frac{1}{360}\left(H^2 + \frac{\beta+1}{7}H + \frac{10}{21}\beta\right)W^{VI} + \frac{1}{135}\left[H + \frac{3}{14}(\beta+1)\right]W^{IV} - \frac{1}{2835}W''\right\}\Xi^9 +$$
$$+ \left\{\frac{1}{30240}H^2(H^2+4\beta)W^{VIII} - \frac{7}{10800}\left[H^3 - \frac{180}{49}\beta H + \frac{30}{7}\beta(\beta+1)\right]W^{VI} + \frac{17}{11340}\left(H^2 + \frac{162}{85}\beta\right)W^{IV} - \frac{1}{14175}HW''\right\}\Xi^{10} +$$
$$+ \sqrt{\beta}\left\{-\frac{1}{798336}H^4W^{X} + \frac{1}{10395}H\left(H^2 + \frac{\beta+1}{8}H + \beta\right)W^{VIII} - \frac{1}{110}\left[\frac{7}{60}H^2 + \frac{\beta+1}{42}H + \frac{1}{252}(\beta^2 + 26\beta + 1)\right]W^{VI} + \frac{1}{110}\left(\frac{68}{567}H + \frac{\beta+1}{35}\right)W^{IV} - \frac{2}{155925}W''\right\}\Xi^{11} + \ldots = \Psi + W + \Sigma + Q,$$
$$\Xi = \xi - \xi_0. \tag{3.9}$$

The bracketed expression with which formula (3.9) begins is a harmonic function and determines the focusing electrodes maintaining an elliptic beam [9] ($W = 0$). We denote it by ψ. Figures 1–10 show the results of calculation of the equipotential surfaces for several values of β and $\lambda < 1$, where the variation of the z-component of the velocity is described by a curtate cycloid. In this case $d/a = (2)^{1/2}$, $0 \le z \le \pi$, and the picture is symmetric relative to $z = 0$ and $z = \pi$. Figures 1–4 are for cylindrical beams. We recall that the quantity γ defines the region of variation of the potential on the boundary with movement along the z-axis. The criterion of accuracy was the value of the discrepancy modulus $N(\xi, \eta, z; n) = \Delta\varphi(\xi, \eta, z; n)$. At each point the number of terms of series (3.9) in the range $n = 0, \ldots, 11$,

$$\varphi(n) = \sum_{k=0}^{n} \vartheta_k \Xi^k,$$

$$\vartheta_0 = \Psi + W, \quad \vartheta_1 = 0, \tag{3.10}$$

was automatically chosen so that the discrepancy was a minimum. The greatest distance from the beam boundary was determined by the requirement $|N| = 0.2$. The function $n(R, z)$ for $\beta = 1$, $\gamma = 0.4$ is shown in Table 1. Here $l = R_f/\Delta R$ is the number of steps, counted from $R = 1$, up to which we must use $n(R_f/\Delta R)$ terms of the expansion. For instance, for $t = 20°$ and $l = 1$ we have $n = 8$, for $l = 2, \ldots, 10$ we take $n = 11$, and for $l = 11, \ldots, 28$, $n = 7$; here $\Delta R = 0.025$. The requirement $|N| \ll 1$ when $\varphi \sim 1$ is fairly strict. Table 2 gives the potential values calculated from formula (3.10) for different n and the corresponding discrepancy values.

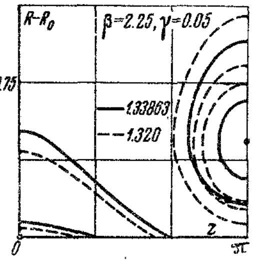

Fig. 10

We note that the presence of a kink in the curve $W = W(z)$ when $t = \pi/2$ enables us to write an exact expression for the potential $\varphi_{ex}(\xi, \eta)$ in the plane $z = \pi/2 - \gamma$

$$\varphi_{ex} = \alpha\left[\frac{\beta+1}{2\beta} + \frac{\beta^2+1}{\beta^{3/2}}\Xi - \frac{\beta-1}{2\beta}\cos 2\eta\left(\operatorname{ch} 2\Xi + \frac{\beta+1}{\beta^{1/2}}\operatorname{sh} 2\Xi\right)\right] + 1 = \Psi + 1. \tag{3.11}$$

This makes it possible to estimate the error at the points of maximum distance from the beam boundary; when $\gamma = 0.4$ it does not exceed 3% for potential and 5% for R. As is to be expected, the convergence of the expansions becomes worse with increase in β and γ.

The higher-order derivatives in (3.9) were determined analytically by differentiation of the equation for W; this entailed the use of its first integral

$$W'' = 2(W^{-1/2} - 1),$$
$$W'^2 = 8W^{1/2} - 4W + 4(\gamma^2 - 1).$$

Figures 5–9 show the curves obtained from the intersection of the surfaces φ = const by the half-planes $\psi = 0$ and $\psi = 90°$. Here R and ψ are ordinary polar coordinates with the pole at $x = y = 0$ and angle measured from the semimajor axis of the ellipse-boundary

$$R_0 = \beta^{1/2}[1 + (\beta - 1)\sin^2\psi]^{-1/2}. \tag{3.12}$$

When $\psi = 0$, the potential at each of the cross sections z = const varies nonmonotonically, attaining a maximum value at some distance from $R = R_0$. This leads to the appearance of closed curves, finally contracting to a point. When $\psi = 90°$, the potential maximum presumably occurs at large R.

Figure 10 shows the evolution of the equipotentials $\varphi = 1.33863$ and $\varphi = 1.320$ from $\psi = 0$ to $\psi = 90°$ ($\psi = 0, 5, 10, 30, 60°$). The first of them at $\psi = 0$ degenerates into a point. With increase in ψ the curves move from right to left. Between 10 and 30° the equipotentials split [the second (upper) branch is not shown in the figure]; R_0 varies in accordance with formula (3.12).

Figure 1 shows the change in the discrepancy for $\beta = 1$, $\gamma = 0.05$ and $\gamma = 0.4$, $t = 0$, 80, and 180°; a logarithmic scale is used for $|N|$.

We note that a flow with periodic variation of v_z cannot be realized with a two-electrode system, at least with the potentials indicated in the figures. However, this is also a feature of an annular beam in the acceleration region $0 \le \psi \le 60°$.

3.3. **Case of nonmonotonic variation of potential in flow between parallel planes.** In the dimensionless variables used in [10], the Cauchy

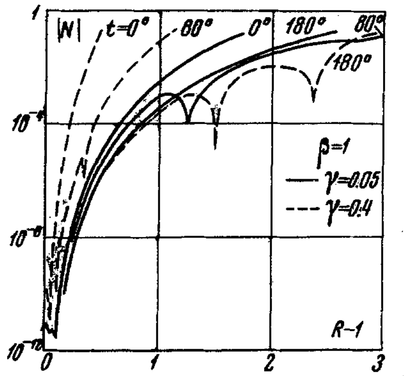

Fig. 11

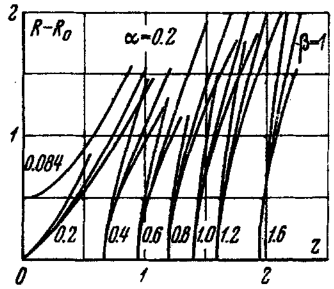

Fig. 12

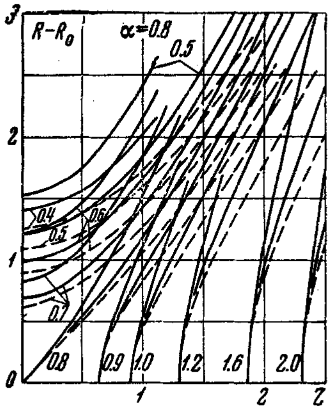

Fig. 13

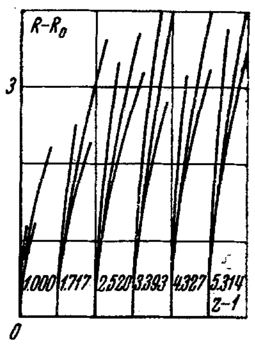

Fig. 14

Table 1

$\beta = 1,\ \gamma = 0.4$

t, °	$n\,(R_f/\Delta R)$	t, °	$n\,(R_f/\Delta R)$
0	11 (18), 10 (19)	100	9 (1), 11 (64), 7 (65), 11 (73)
10	11 (20), 10 (22)	110	9 (1), 11 (12), 8(13), 11 (72), 7 (81)
20	8 (1), 11 (10), 7 (28)	120	9 (1), 11 (23), 8 (26), 11 (85), 7 (96)
30	9 (1), 11 (15), 5 (29)	130	9 (1), 11 (39), 8 (47), 11 (98)
40	9 (1), 11 (18), 4 (29)	140	9 (1), 11 (61), 8 (70), 11 (104)
50	8 (1), 11 (3), 10 (8), 11 (27), 9 (35), 3 (40), 9 (42), 3 (46)	150	10 (1), 9 (3), 11 (10), 9 (14), 11 (91), 8 (94), 11 (109)
60	8 (1), 10(2), 11 (3), 9 (5), 11 (27), 8 (49), 2 (54)	160	10 (1), 9 (3), 11 (28), 9 (47), 11 (109), 8 (120)
70	8 (1), 10 (2), 11 (18), 8 (30), 2 (33), 8 (40)	170	10 (1), 9 (3), 11 (43), 9 (65), 11 (116)
80	8 (1), 10 (2), 11 (12), 8 (16), 11 (45)	180	10 (1), 8 (2), 9 (3), 10 (4), 11 (48), 9 (70), 11 (117)

Table 2

$\beta = 1,\ \gamma = 0.4$

$\|N(n)\|$	$\varphi(n)$	$\|N(n)\|$	$\varphi(n)$
$t = 0,\ R = 1.15$		$t = 180°,\ R = 1.6$	
1.76	0.679880970	1.46	2.44500181
0.510	0.666858704	0.950	2.50811707
0.144	0.665645360	0.432	2.52789334
$0.354 \cdot 10^{-1}$	0.665462433	0.159	2.53296422
$0.869 \cdot 10^{-2}$	0.665435748	$0.516 \cdot 10^{-1}$	2.53415636
$0.213 \cdot 10^{-2}$	0.665431355	$0.146 \cdot 10^{-1}$	2.53442748
$0.536 \cdot 10^{-3}$	0.665430592	$0.340 \cdot 10^{-2}$	2.53448583
$0.137 \cdot 10^{-3}$	0.665430448	$0.580 \cdot 10^{-3}$	2.53449690
$0.354 \cdot 10^{-4}$	0.665430420	$0.411 \cdot 10^{-4}$	2.53449856
$0.980 \cdot 10^{-5}$	0.665430414	$0.143 \cdot 10^{-4}$	2.53449869
$0.336 \cdot 10^{-5}$	0.665430413	$0.589 \cdot 10^{-5}$	2.53449868
$t = 100°,\ R = 1.6$		$t = 180°,\ R = 1.9$	
0.332	1.62874487	2.06	2.53092694
0.421	1.64309211	1.65	2.64863449
0.367	1.64758761	0.979	2.69900185
0.235	1.65021314	0.482	2.71663879
0.108	1.65159170	0.207	2.72230122
$0.353 \cdot 10^{-1}$	1.65211406	$0.771 \cdot 10^{-1}$	2.72405981
$0.640 \cdot 10^{-2}$	1.65225890	$0.236 \cdot 10^{-1}$	2.72457672
$0.114 \cdot 10^{-2}$	0.65228643	$0.514 \cdot 10^{-2}$	2.72471065
$0.163 \cdot 10^{-2}$	1.65228767	$0.359 \cdot 10^{-3}$	2.72473796
$0.792 \cdot 10^{-3}$	1.65228576	$0.291 \cdot 10^{-3}$	2.72474109
$0.193 \cdot 10^{-3}$	1.65228467	$0.130 \cdot 10^{-3}$	2.72474062

Table 3

$\lvert N(n) \rvert$	$\varphi(n)$	$\lvert N(n) \rvert$	$\varphi(n)$
$\beta=1,\ \alpha=0.2$			
$Z=0,\ R=1.2$		$Z=1.602,\ R=2$	
1.43	0.200000000	1.62	1.20000000
0.496	0.183482339	1.05	1.10253520
0.141	0.181474656	0.409	1.05749690
$0.388\cdot10^{-1}$	0.181177953	$0.488\cdot10^{-1}$	1.04357033
$0.120\cdot10^{-1}$	0.181131443	$0.449\cdot10^{-1}$	1.04110855
$0.399\cdot10^{-2}$	0.181121960	$0.285\cdot10^{-1}$	1.04123565
$0.139\cdot10^{-2}$	0.181119768	$0.228\cdot10^{-2}$	1.04148818
$0.494\cdot10^{-3}$	0.181119197	$0.785\cdot10^{-2}$	1.04156454
$0.178\cdot10^{-3}$	0.181119041	$0.691\cdot10^{-2}$	1.04155977
$0.650\cdot10^{-4}$	0.181118996	$0.305\cdot10^{-2}$	1.04154283
$0.239\cdot10^{-4}$	0.181118983	$0.242\cdot10^{-3}$	1.04153294
$\beta=25,\ \psi=0,\ \alpha=0.2$			
$Z=0,\ R=5.2$		$Z=1.602,\ R=5.6$	
3.02	0.200000000	2.99	1.20000000
1.60	0.189400000	1.48	1.17843383
0.186	0.184239388	$0.795\cdot10^{-1}$	1.15499479
0.113	0.184117198	$0.273\cdot10^{-1}$	1.15431293
$0.431.10^{-1}$	0.184040478	$0.684\cdot10^{-1}$	1.15402944
$0.142\cdot10^{-1}$	0.184018797	$0.178\cdot10^{-1}$	1.15419896
$0.600\cdot10^{-2}$	0.184014476	$0.126\cdot10^{-2}$	1.15424456
$0.264\cdot10^{-2}$	0.184013115	$0.310\cdot10^{-2}$	1.15424824
$0.104\cdot10^{-2}$	0.184012612	$0.177\cdot10^{-2}$	1.15424633
$0.111\cdot10^{-2}$	0.184012074	$0.576\cdot10^{-2}$	1.15423750
$0.136\cdot10^{-2}$	0.184012023	$0.619\cdot10^{-2}$	1.15423710

conditions at the beam boundary are written as follows:

$$Z = z - \sigma = \mp (\varphi_0^{1/2} + 2\alpha^{1/2}) \sqrt{\varphi_0^{1/2} - \alpha^{1/2}},$$

$$\sigma = (1 + 2\alpha^{1/2}) \sqrt{1 - \alpha^{1/2}}, \quad \varphi_1 = 0. \qquad (3.13)$$

Relationship $\varphi_0 = \varphi_0(z)$ is specified implicitly: $\alpha = \varphi_0(\sigma)$ is the minimum value of the potential $\alpha \le \varphi_0$. The solution is given by formulas (3.5) without any changes. Higher derivatives were found by differentiation of the equation for φ_0 with the use of its first integral

$$\varphi_0'' = {}^4/_9 \varphi_0^{-1/2}, \qquad \varphi_0'^2 = {}^{16}/_9 (\varphi_0^{1/2} - \alpha^{1/2}).$$

The potential was calculated as in the previous case. The results are given in Figs. 12, 13. In each bundle of three curves to which some value of potential is ascribed, the middle one corresponds to the value $\beta = 1$ (cylindrical beam), and the other two to $\beta = 25$. The curve with the greater slope corresponds to the section $\psi = 0$, and the one with the lesser slope to $\psi = 90°$. In Fig. 13, the latter is shown by dashed lines. The cited results can be used to construct periodic focusing systems for beams of circular and elliptic cross section.

Table 3 gives the values of $\varphi(n)$ and $|N(n)|$ for $\alpha = 0.2$ and different R, z. We note that reduction of α hinders the convergence of the series.

3.4. **Accelerating electrodes for emission limited by space charge.** At the beam boundary $\xi = \xi_0$ we have

$$\varphi_0 = z^{4/3}, \qquad \varphi_1 = 0. \qquad (3.14)$$

Some of the terms in expansion (3.5) can be summed. As a result we obtain

$$\varphi = (z^2 + H\Xi^2)^{2/3} \cos {}^4/_3 \operatorname{arc\,tg}[(\sqrt{H}\Xi) / z] + Q,$$

where Q is the polynomial in (3.9).

The results of calculating the accelerating electrodes are given in Fig. 14. As in the two previous figures, this shows equipotential curves corresponding to cylindrical ($\beta = 1$) and elliptic beams ($\beta = 25$, $\psi = 0$ and $\psi = 90°$); the curves with the greatest slope correspond to the case $\beta = 25$, $\psi = 0$.

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